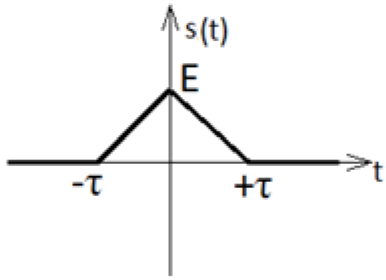
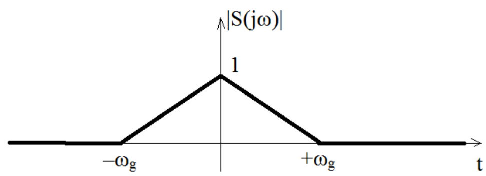
## Domaći zadatak

1. Pronaći Furijeovu transformaciju signala datog na slici direktnom prinmenom izraza za aperiodičnu furijeovu transformaciju.




2. Ako je spektralna gustina amplituda signala s(t) oblika kao na slici pronaći:
   a. Kružnu frekvenciju $\omega_1$ tako da četvrtinu ukupne energije signala ostvaruju komponente iz opsega $|\omega| < \omega_1$,

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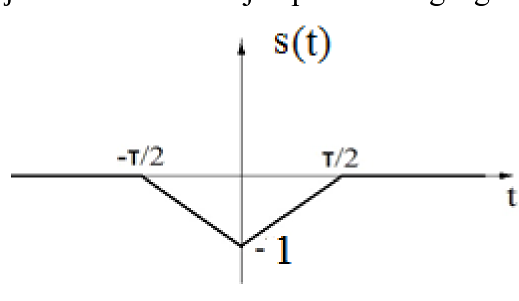
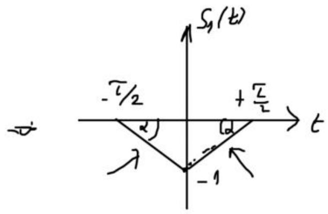
ZADACI SA VEŽBI.

1. Pronaći Furijeovu transformaciju signala prikazanog na slici 1. direktnom primenom izraza za Furijeovu transformaciju aperiodičnog signala.




Slika 1.

REŠENJE:




$$S_1(j\omega)=\int_{-\infty}^{+\infty} s_1(t)\cdot e^{-j\omega t}\,dt=\int_{-\frac{\tau}{2}}^{0}\left(-1-\frac{2}{\tau}t\right)e^{-j\omega t}\,dt+\int_{0}^{\tau/2}\left(-1+\frac{2}{\tau}t\right)e^{-j\omega t}\,dt=I+II$$

$$s_1(t)=\begin{cases}0, & t>|\tau/2|\\ -1-\frac{2}{\tau}t, & -\frac{\tau}{2}<t<0\\ -1+\frac{2}{\tau}t, & 0<t<+\frac{\tau}{2}\end{cases}$$

$$I=\int_{-\frac{\tau}{2}}^{0}\left(-1-\frac{2}{\tau}t\right)e^{-j\omega t}\,dt=\left|\begin{array}{l}\text{PARCIJALNA INT.}\\ u=-1-\frac{2}{\tau}\cdot t \Rightarrow du=-\frac{2}{\tau}dt\\ dv=e^{-j\omega t}dt \Rightarrow v=\int dv=\int e^{-j\omega t}dt=\frac{1}{-j\omega}e^{-j\omega t}\end{array}\right|=$$

$$\int_a^b u\,dv=u\cdot v\Big|_a^b-\int_a^b v\,du$$

$$=\left(-1-\frac{2}{\tau}t\right)\cdot\frac{1}{-j\omega}\cdot e^{-j\omega t}\Big|_{-\frac{\tau}{2}}^{0}-\int_{-\frac{\tau}{2}}^{0}\frac{1}{-j\omega}\cdot e^{-j\omega t}\left(-\frac{2}{\tau}\right)\cdot dt=$$

$$=\left(\left(-1-\frac{2}{\tau}\cdot 0\right)\cdot\frac{1}{-j\omega}\cdot e^{0}-\left(-1-\frac{2}{\tau}\left(-\frac{\tau}{2}\right)\right)\cdot\frac{1}{-j\omega}\cdot e^{-j\omega\left(-\frac{\tau}{2}\right)}\right)-\frac{2}{j\omega\tau}\int_{-\frac{\tau}{2}}^{0}e^{-j\omega t}\,dt=$$

$$=\left(\frac{1}{j\omega}\right)+\frac{2}{j\omega\tau}\cdot\frac{1}{j\omega}e^{-j\omega t}\Big|_{-\frac{\tau}{2}}^{0}=\frac{1}{j\omega}+\frac{2}{j^2\omega^2\tau}\left(1-e^{j\omega\frac{\tau}{2}}\right)=$$

$$=\frac{1}{j\omega}-\frac{2}{\omega^2\tau}\left(1-e^{j\omega\frac{\tau}{2}}\right)=\frac{1}{j\omega}-\frac{2}{\omega^2\tau}+\frac{2}{\omega^2\tau}e^{j\omega\frac{\tau}{2}}$$

$$II=\int_{0}^{-\tau/2}\left(-1+\frac{2}{\tau}t\right)e^{-j\omega t}\,dt=\ldots=-\frac{1}{j\omega}-\frac{2}{\omega^2\tau}+\frac{2}{\omega^2\tau}e^{-j\omega\frac{\tau}{2}}$$

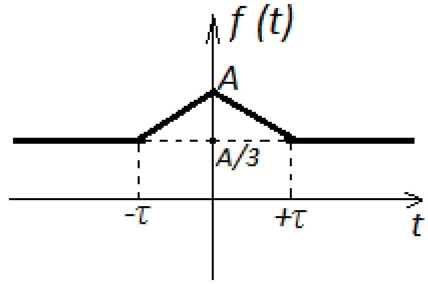
$$I+II=\cancel{\frac{1}{j\omega}}-\frac{2}{\omega^2\tau}+\frac{2}{\omega^2\tau}e^{j\omega\frac{\tau}{2}}-\cancel{\frac{1}{j\omega}}-\frac{2}{\omega^2\tau}+\frac{2}{\omega^2\tau}e^{-j\omega\frac{\tau}{2}}=$$

$$= -\frac{4}{\omega^2\tau} + \frac{2}{\omega^2\tau}\left(e^{j\omega\frac{\tau}{2}} + e^{-j\omega\frac{\tau}{2}}\right) = -\frac{4}{\omega^2\tau} + \frac{2}{\omega^2\tau}\cdot 2\cos\frac{\omega\tau}{2} = -\frac{4}{\omega^2\tau}\left(1-\cos\frac{\omega\tau}{2}\right)$$

$$= -\frac{4}{\omega^2\tau}\cdot 2\sin^2\left(\frac{\omega\tau}{4}\right)\cdot\frac{\tau}{\tau}\cdot\frac{4}{4} = -2\cdot\tau\,\frac{\sin^2\frac{\omega\tau}{4}}{\frac{\omega^2\tau^2}{2^2}} = -\cancel{2}\,\frac{\tau}{2}\underbrace{\left(\frac{\sin\frac{\omega\tau}{4}}{\frac{\omega\tau}{4}}\right)^2}$$

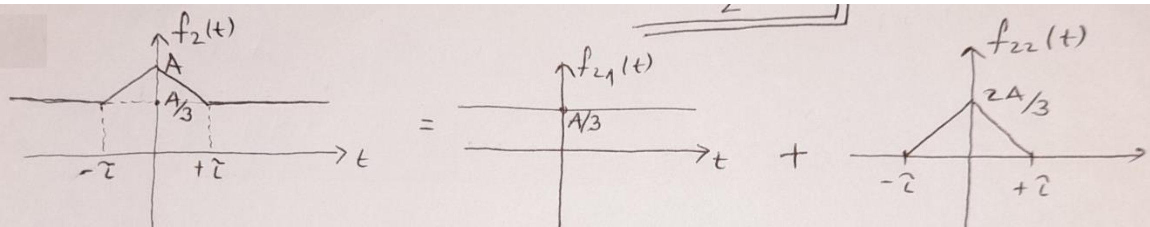
$$1-\cos x = 2\sin^2\frac{x}{2}$$

2. Pronaći spektar signala sa slike 2.



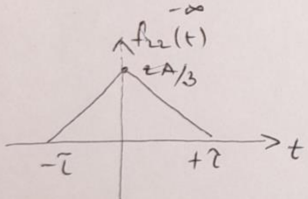
Slika 2.

REŠENJE:

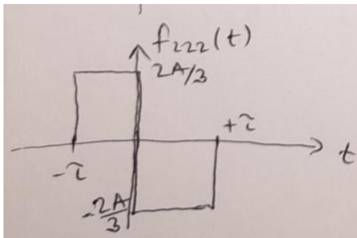



$$F_{21}(j\omega) = \int_{-\infty}^{+\infty}\frac{A}{3}e^{-j\omega t}\,dt = \frac{A}{3}\int_{-\infty}^{+\infty}\cos(\omega t)\,dt = 2\pi\cdot\frac{A}{3}\cdot\delta(\omega)$$

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\cos(\omega t)\,d\omega \qquad \delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\cos(\omega t)\,dt \Rightarrow \int_{-\infty}^{+\infty}\cos(\omega t)\,dt = 2\pi\delta(\omega)$$

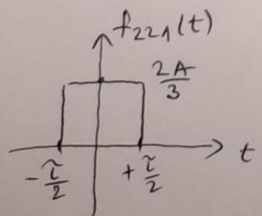



$$f_{22}(t) = \begin{cases} 0, & |t| > \tau \\ \frac{2A}{3} + \frac{2A}{3\tau}\cdot t, & -\tau < t < 0 \\ \frac{2A}{3} - \frac{2A}{3\tau}\cdot t, & 0 < t < +\tau \end{cases}$$




$$\boxed{f_{222}(t) = \tau \cdot \frac{d f_{22}(t)}{dt} = \begin{cases} 0, & |t| > \tau \\ \frac{2A}{3}, & -\tau < t < 0 \\ -\frac{2A}{3}, & 0 < t < \tau \end{cases}}$$

POMOĆNI SIGNALI




$$\boxed{f_{222}(t) = f_{221}\left(t + \frac{\tau}{2}\right) - f_{221}\left(t - \frac{\tau}{2}\right)}$$

$$F_{221}(j\omega) = E \cdot \tau \cdot \frac{\sin \frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}$$

$$\boxed{\begin{aligned} f(t) &\rightarrow F(j\omega) \\ f(t+T_0) &\rightarrow e^{j\omega T_0} \cdot F(j\omega) \end{aligned}}$$ PRAVILO

$$f_{222}(t) = f_{221}\left(t + \frac{\tau}{2}\right) - f_{221}\left(t - \frac{\tau}{2}\right)$$

$$F_{222}(j\omega) = F_{221}(j\omega) \cdot e^{j\omega\frac{\tau}{2}} - F_{221}(j\omega) \cdot e^{-j\omega\frac{\tau}{2}} =$$

$$= F_{221}(j\omega)\left(e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right) \cdot \frac{2j}{2j} = j2 \cdot F_{221}(j\omega) \cdot \sin\left(\frac{\omega\tau}{2}\right) =$$

$$= j2 \cdot \frac{2}{3} A \cdot \tau \, \frac{\sin\left(\omega\frac{\tau}{2}\right)}{\frac{\omega\tau}{2}} \cdot \sin\left(\frac{\omega\tau}{2}\right) = j\frac{4}{3} A\tau \, \frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}$$

$$\frac{d f_{22}(t)}{dt} = \frac{1}{\tau} f_{222}(t)$$

$$\boxed{\begin{aligned} f(t) &\rightarrow F(j\omega) \\ \frac{df(t)}{dt} &\rightarrow j\omega \cdot F(j\omega) \end{aligned}}$$ PRAVILO

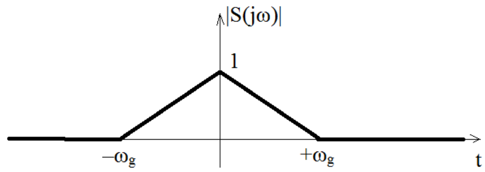
$$j\omega \cdot F_{22}(j\omega) = \frac{1}{\tau} \cdot F_{222}(j\omega)$$

$$F_{22}(j\omega) = \frac{1}{j\omega\tau} \cdot F_{222}(j\omega) = \frac{1}{j\omega\tau} \cdot j\frac{4}{3} A \cdot \tau \cdot \frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} =$$

$$= \frac{2}{3} A\tau \left(\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}\right)^2$$

$$F_2(j\omega) = F_{21}(j\omega) + F_{22}(j\omega) = 2\pi \frac{A}{3}\delta(\omega) + \frac{2}{3}A\tau\left(\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}\right)^2$$

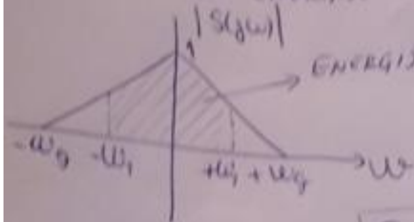
3. Ako je spektralna gustina amplituda signala s(t) oblika kao na slici 3. pronaći:
   a. Kružnu frekvenciju $\omega_1$ tako da polovinu ukupne energije signala ostvaruju komponente iz opsega $|\omega| < \omega_1$,
   b. Talasni oblik signala ukoliko je spektralna gustina faza jednaka nuli i predstaviti ga grafički.




Slika 3.

REŠENJE:

POTREBNO JE NAĆI KRUŽNU FREKVENCIJU $\omega_1$ TAKO DA POLOVINU UKUPNE ENERGIJE SIGNALA OSTVARUJU KOMPONENTE IZ OPSEGA $|\omega| < \omega_1$




ENERGIJA OVOG DELA JE $\frac{E_{UK}}{2}$

~~UKUPNU~~ UKUPNU ENERGIJU SIGNALA ODREĐUJEMO PREKO PARSEVALOVE TEOREME:

$$\boxed{E_{UK} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |S(j\omega)|^2 d\omega}$$

MATEMATIČKI ZAPIŠEMO SPEKTRALNU GUSTINU AMPLITUDA:

$$|S(j\omega)| = \begin{cases} 0, & -\infty < \omega < -\omega_g \\ 1 + \frac{\omega}{\omega_g}, & -\omega_g < \omega < 0 \\ 1 - \frac{\omega}{\omega_g}, & 0 < \omega < +\omega_g \\ 0, & +\omega_g < \omega < +\infty \end{cases} = \begin{cases} 0, & |\omega| > \omega_g \\ 1 - \frac{|\omega|}{\omega_g}, & |\omega| < \omega_g \end{cases}$$

NA OSNOVU PRETHODNOG NASTAVLJAMO DA TRAŽIMO ENERGIJU SIGNALA

$$E_{UK} = \frac{1}{2\pi}\int_{-\omega_g}^{+\omega_g} \left|1 - \frac{|\omega|}{\omega_g}\right|^2 d\omega \underset{\substack{\text{PARNA FUNKCIJA}\\ \text{NA SIMETRIČNOM}\\ \text{INTERVALU}}}{=} \frac{1}{2\pi}\cdot 2\cdot\int_0^{+\omega_g}\left|1 - \frac{|\omega|}{\omega_g}\right|^2 d\omega = \frac{1}{\pi}\int_0^{+\omega_g}\left|1 - \frac{|\omega|}{\omega_g}\right|^2 d\omega =$$

$$= \left| \begin{array}{l} \text{SMENA:} \\ 1-\frac{\omega}{\omega_g} = x \Rightarrow -\frac{d\omega}{\omega_g} = dx \\ \text{DONJA GRANICA: } 1-\frac{0}{\omega_g}=1 \\ \text{GORNJA GRANICA: } 1-\frac{\omega_g}{\omega_g}=0 \end{array}\right| = \frac{1}{\pi}\int_1^0 x^2\cdot(-\omega_g)\cdot dx = -\frac{\omega_g}{\pi}\cdot\int_1^0 x^2 dx =$$

$$= -\frac{\omega_g}{\pi}\cdot\frac{x^3}{3}\Big|_1^0 = -\frac{\omega_g}{3\pi}\left(0^3-1^3\right) = \boxed{+\frac{\omega_g}{3\pi}}$$ UKUPNA ENERGIJA SIGNALA

SADA TREBA DA ODREDIMO $\omega_1$ TAKO DA U TOM DELU BUDE ~~SKONCENTRISANA~~ SKONCENTRISANA POLOVINA ENERGIJE, ALI PRVO DA NAĐEMO OPŠTI OBRAZAC ZA ENERGIJU U GRANICAMA OD $-\omega_1$ DO $+\omega_1$

$$E_1 = \frac{1}{2\pi}\int_{-\omega_1}^{+\omega_1}\left|1-\frac{|\omega|}{\omega_g}\right|^2 d\omega = \frac{1}{2\pi}\cdot 2\cdot\int_0^{\omega_1}\left|1-\frac{|\omega|}{\omega_g}\right|^2 d\omega = \left| \begin{array}{l} 1-\frac{\omega}{\omega_g} = x \Rightarrow -\frac{d\omega}{\omega_g} = dx \\ \text{DONJA GRANICA: } 1-\frac{0}{\omega_g}=1 \\ \text{GORNJA GRANICA: } 1-\frac{\omega_1}{\omega_g} \end{array}\right| =$$

$$= \frac{1}{\pi}\int_1^{1-\frac{\omega_1}{\omega_g}} x^2\cdot(-\omega_g)\,dx = -\frac{\omega_g}{\pi}\cdot\int_1^{1-\frac{\omega_1}{\omega_g}} x^2 dx = -\frac{\omega_g}{\pi}\,\frac{x^3}{3}\Big|_1^{1-\frac{\omega_1}{\omega_g}} =$$

$$= -\frac{\omega_g}{3\pi}\left(\left(1-\frac{\omega_1}{\omega_g}\right)^3-1\right)$$

$$E_1 = \frac{E_{UK}}{2} \Rightarrow -\frac{\omega_g}{3\pi}\left(\left(1-\frac{\omega_1}{\omega_g}\right)^3-1\right) = \frac{\frac{\omega_g}{3\pi}}{2}$$

$$-\frac{\omega_g}{3\pi}\left(\left(1-\frac{\omega_1}{\omega_g}\right)^3-1\right) = \frac{\omega_g}{6\pi} \quad \Big| : \frac{\omega_g}{3\pi}$$

$$-\left(1-\frac{\omega_1}{\omega_g}\right)^3 + 1 = \frac{1}{2}$$

$$-\left(1-\frac{\omega_1}{\omega_g}\right)^3 = \frac{1}{2} - 1 = -\frac{1}{2} \quad \Big|\cdot(-1)$$

$$\left(1-\frac{\omega_1}{\omega_g}\right)^3 = \frac{1}{2} \quad \Big|\sqrt[3]{\phantom{x}}$$

$$\sqrt[3]{\left(1-\frac{\omega_1}{\omega_g}\right)^3} = \sqrt[3]{\frac{1}{2}}$$

$$\left(1-\frac{\omega_1}{\omega_g}\right) = \frac{1}{\sqrt[3]{2}} \Rightarrow \frac{\omega_1}{\omega_g} = 1 - \frac{1}{\sqrt[3]{2}}$$

$$\boxed{\omega_1 = \omega_g\cdot\left(1-\frac{1}{\sqrt[3]{2}}\right)}$$

b)

b) Spektar $S(j\omega)$ se može prikazati:

$$S(j\omega) = \underbrace{|S(j\omega)|}_{\text{spektralna gustina amplituda}} \cdot e^{j\underbrace{\theta(\omega)}_{\text{spektralna gustina faza}}}$$

, u zadatku je rečeno da je spektralna gustina faza jednaka nuli, pa zbog toga sledi:

$$S(j\omega) = |S(j\omega)|$$

Pošto se traži talasni oblik funkcije moramo da primenimo inverznu Furijeovu transformaciju

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_g}^{+\omega_g}\left(1-\frac{|\omega|}{\omega_g}\right)e^{j\omega t}d\omega =$$

$$= \frac{1}{2\pi}\int_{-\omega_g}^{0}\left(1+\frac{\omega}{\omega_g}\right)e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{+\omega_g}\left(1-\frac{\omega}{\omega_g}\right)e^{j\omega t}d\omega =$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\omega_g}^{0}e^{j\omega t}d\omega} + \frac{1}{2\pi}\int_{-\omega_g}^{0}\frac{\omega}{\omega_g}e^{j\omega t}d\omega + \underbrace{\frac{1}{2\pi}\int_{0}^{+\omega_g}e^{j\omega t}d\omega} - \frac{1}{2\pi}\int_{0}^{+\omega_g}\frac{\omega}{\omega_g}e^{j\omega t}d\omega =$$

$$= \frac{1}{2\pi}\int_{-\omega_g}^{+\omega_g}e^{j\omega t}d\omega + \underbrace{\frac{1}{2\pi\omega_g}\int_{-\omega_g}^{0}\omega e^{j\omega t}d\omega} - \underbrace{\frac{1}{2\pi\omega_g}\int_{0}^{+\omega_g}\omega e^{j\omega t}d\omega} = *$$

PAUZA

Ova dva integrala mora da se rade preko parcijalne integracije.
Rešićemo jedan opšti slučaj za granice od a do b pa ćemo to posle direktno da primenimo.

STRANA 4.

$$I = \int_a^b \omega\cdot e^{j\omega t}d\omega = \left| \begin{array}{l} u=\omega \Rightarrow du = d\omega \\ dv = e^{j\omega t}d\omega \Rightarrow v = \int dv = \int e^{j\omega t}d\omega = \frac{1}{jt}e^{j\omega t} \end{array}\right| =$$

$$= \omega\cdot\frac{1}{jt}e^{j\omega t}\Big|_a^b - \int_a^b \frac{1}{jt}\cdot e^{j\omega t}d\omega = \left(b\cdot\frac{1}{jt}\cdot e^{jbt} - a\cdot\frac{1}{jt}e^{jat}\right) - \frac{1}{jt}\int_a^b e^{j\omega t}d\omega = \frac{1}{jt}\left(b\cdot e^{jbt} - a\cdot e^{jat}\right) - \frac{1}{jt}\cdot\frac{1}{jt}e^{j\omega t}\Big|_a^b =$$

$$= \frac{1}{jt}\left(b\cdot e^{jbt} - a\cdot e^{jat}\right) - \frac{1}{j^2t^2}\left(e^{jbt} - e^{jat}\right) =$$

$$= \frac{1}{jt}\left(b\cdot e^{jbt} - a\cdot e^{jat}\right) + \frac{1}{t^2}\left(e^{jbt} - e^{jat}\right)$$

... rešimo integrale, samo

NA OSNOVU OVOG REŠENJA DIREKTNO REŠIMO INTEGRALE, TAMO GDE SMO STALI:

$$(*) = \frac{1}{2\pi j t} e^{j\omega t}\Big|_{-\omega_g}^{+\omega_g} + \frac{1}{2\pi\omega_g}\left(\frac{1}{jt}\left(0\cdot e^{j0t} - (-\omega_g)\cdot e^{-j\omega_g t}\right) + \frac{1}{t^2}\left(e^{j0t} - e^{-j\omega_g t}\right)\right)$$

$$-\frac{1}{2\pi\omega_g}\left(\frac{1}{jt}\left(\omega_g\cdot e^{j\omega_g t} - 0\cdot e^{j0t}\right) + \frac{1}{t^2}\left(e^{j\omega_g t} - e^{j0t}\right)\right) =$$

$$= \frac{1}{2\pi j t}\left(e^{j\omega_g t} - e^{-j\omega_g t}\right) + \frac{1}{2\pi\omega_g}\left(\frac{1}{jt}\cdot\omega_g\cdot e^{-j\omega_g t} + \frac{1}{t^2} - \frac{1}{t^2}e^{-j\omega_g t}\right) -$$

$$\frac{1}{2\pi\omega_g}\left(\frac{1}{jt}\omega_g e^{j\omega_g t} + \frac{1}{t^2}e^{j\omega_g t} - \frac{1}{t^2}\right) = \cancel{\frac{1}{2\pi j t}e^{j\omega_g t}} - \cancel{\frac{1}{2\pi j t}e^{-j\omega_g t}} +$$

$$+ \cancel{\frac{1}{2\pi\omega_g j t}\omega_g e^{-j\omega_g t}} + \frac{1}{2\pi\omega_g t^2} - \frac{1}{2\pi\omega_g t^2}e^{-j\omega_g t} - \cancel{\frac{1}{2\pi\omega_g j t}\omega_g e^{j\omega_g t}} -$$

$$\frac{1}{2\pi\omega_g t^2}e^{j\omega_g t} + \frac{1}{2\pi\omega_g t^2} = \frac{1}{2\pi\omega_g t^2} - \frac{1}{2\pi\omega_g t^2}e^{-j\omega_g t} - \frac{1}{2\pi\omega_g t^2}e^{j\omega_g t} + \frac{1}{2\pi\omega_g t^2} =$$

$$= \frac{1}{2\pi\omega_g t^2}\left(1 - e^{-j\omega_g t} - e^{j\omega_g t} + 1\right) = \frac{1}{2\pi\omega_g t^2}\left(2 - \left(e^{j\omega_g t} + e^{-j\omega_g t}\right)\right) =$$

$$= \frac{1}{2\pi\omega_g t^2}\left(2 - 2\cos\omega_g t\right) = \frac{2}{2\pi\omega_g t^2}\left(1 - \cos\omega_g t\right) =$$

$$= \left(\frac{1}{\pi\omega_g t^2}\cdot 2\cdot\sin^2\frac{\omega_g t}{2}\right)\cdot\frac{\omega_g}{\omega_g}\cdot\frac{2}{2} = \frac{\omega_g}{2\pi}\frac{\sin^2\omega_g t/2}{\frac{\omega_g^2 t^2}{2^2}} = \frac{\omega_g}{2\pi}\left(\frac{\sin\frac{\omega_g t}{2}}{\frac{\omega_g t}{2}}\right)^2$$

$$s(t) = \frac{\omega_g}{2\pi}\left(\frac{\sin\frac{\omega_g t}{2}}{\frac{\omega_g t}{2}}\right)^2$$

nule za $\frac{\omega_g t}{2} = k\pi$, $k = \pm 1, \pm 2 \ldots$

$t = \frac{2k\pi}{\omega}$, $k = \pm 1, \pm 2 \ldots$